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GLOBALLY OPTIMAL PATHS IN THE NONCLASSICAL GROWTH MODEL†

R. Amir

Department of Applied Mathematics and Statistics, SUNY at Stony Brook, Stony Brook, NY 11794-3600, U.S.A.

Abstract—We consider the nonclassical optimal growth model, consisting of convex preferences and a convex–concave production function. We give necessary and sufficient conditions for a feasible path of one-period consumptions to be globally optimal. In the process, we also characterize locally optimal paths, and present an example which illustrates most points of interest.

The analysis is based on an altered dynamic programming approach.

1. INTRODUCTION

In recent years, optimal growth theorists have been concerned with extending the scope of normative growth theory beyond the classical one-sector model, largely based on the assumption of convexity, both in preferences and in technology. Perhaps the most elaborate outcome of the new focus has been, so far, the nonclassical model of optimal growth. This model rests on the assumptions of convex preferences and allows for increasing returns to scale technology. Thus, the production function is required to be concave only for large input levels.

This new development, initiated in a continuous-time model by Skiba (1978), was subsequently studied by Majumdar and Mitra (1982, 1983) and Dechert and Nishimura (1983) in discrete-time. For a more detailed motivation behind this model, the reader is referred to those studies.

The optimization problem in this model consists of maximizing a concave functional subject to nonconvex set. As a general theory for such problems is not available at this time, one has to resort to nonstandard arguments in trying to take advantage of the particular structure of the problem at hand.

Dechert and Nashimura essentially characterize optimality by the Euler equations, the transversality condition (recall that these are necessary and sufficient in the classical case), and, in addition, by a monotonicity property of optimal paths, that they derive. Subsequently, Amir (1984) and Amir *et al.* (1984) give the equivalent property that the marginal propensity of consumption is always bounded above by unity, as a second-order condition for optimality. An example is also given there, showing among other things that there may be interior local, but not global, maximizers satisfying this second-order condition.

In the present paper, a dynamic programming approach is altered in a way that allows for the analysis of the properties of local maxima, as well as local minima. It turns out that these properties constitute a necessary intermediate step to derive necessary and sufficient conditions for global optimality. First, we show that the monotocity property of optimal paths (or, equivalently, the uniform boundedness of the marginal propensity of consumption by unity) is a necessary condition for local (as well as for global) optimality, and is also sufficient for local optimality, but not for global optimality. Finally we show that the well-known properties of the value function—continuity and monotonicity—are sufficient (along with the above conditions) to guarantee global optimality. In other words, if at any stock level, a local non-global maximizer is selected, a discontinuity in the value function will be observed.

We suggest that the previous literature on this problem has not distinguished between local and global maxima,‡ and consequently has not attempted to derive conditions that uniquely characterize global optimality. This is the major aim of this paper, and we hope to have provided some insight towards a systematic approach to nonconvex dynamic optimization.

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‡Though Skiba (1978) clearly refers to such distinctions.

The paper is organized as follows. Section 2 provides the definitions of all the new concepts within the proposed framework, and the properties of local extrema (maxima and minima), In Section 3, analogous properties are derived for global maxima, culminating in the Main Theorem, which gives necessary and sufficient conditions for globally optimal paths. Section 4 consists of a simple example of a one-period horizon problem which illustrates the possibility of existence of interior local nonglobal maxima, possibility which actually occurs whenever optimal paths are not unique, as argued in the discussion following the statement of the Main Theorem. In Section 5, we suggest that while the dynamics of locally optimal paths and the dynamics of globally optimal paths differ, their asymptotic properties (convergence and stability) are qualitatively the same.

2. THE MODEL AND LOCAL OPTIMALITY CONDITIONS

The one-sector nonclassical optimal growth model can be described as follows: consider a central planner whose objective is to maximize the present value of the utility of consumption over an infinite horizon, i.e.

$$\sum_{t=0}^{\infty} \delta^t u(c_t),$$

where $0 < \delta < 1$ is a fixed discount factor, c_t is consumption at time t, and $u(\cdot)$ is the one-period utility function, satisfying

$$u \in C^1(0, +\infty),$$

$$u'(\cdot) > 0,$$

$$u \text{ is strictly concave.}$$

If x_t denotes output available at period t, the production process is described by

$$x_{t+1} = f(x_t - c_t), \quad x_0 = s, \quad t = 0, 1, \ldots$$

where the production function f satisfies

$$f \in C^1[0, +\infty),$$

$$f'(\cdot) \geqslant 0.$$

There exists $\bar{x} > 0$ such that $f(\bar{x}) = \bar{x}, f(x) < x$ if $x > \bar{x}$ and f is concave on $[\bar{x}, +\infty)$.

Assume further that $u'(0) = +\infty, f(0) = 0$ and $f'(0) \neq 0$, so that no corner solutions prevail (see Amir *et al.*, 1984).

Let

$$V(s) = \max_{\{c_t\}} \sum_{t=0}^{\infty} \delta^t u(c_t) \tag{1}$$

subject to

$$x_{t+1} = f(x_t - c_t), \quad x_0 = s \tag{2}$$

and to

$$0 \leqslant c_t \leqslant x_t, t = 0, 1, \ldots \tag{3}$$

A sequence $\{c_t\}$ is feasible if it satisfies (2)–(3). A maximizer is an interior solution if constraint (3) holds with strict inequalities, and a corner or boundary solution otherwise.

Clearly, for any feasible sequence of consumptions $\{c_t\}$, $x_t \leqslant \max\{x_0, \bar{x}\}$, where x_0 is the initial stock and $\bar{x}$ is the largest fixed point of f. The one-period utilities of consumption are thus uniformly bounded. Let $S = [0, \max\{x_0, \bar{x}\}]$ and $B(S)$ be the space of real-valued functions on S with sup norm. The operator $T: B(S) \to B(S)$

$$v \to \sup_{0 \leqslant c \leqslant x} \{u(c) + \delta v[f(x - c)]\}$$

is a contraction mapping on a complete metric space. Hence there exists a unique $V \in B(S)$ such that†

$$V(x) = \sup_{0 \leqslant c \leqslant x} M(c; x), \tag{4a}$$

where

$$M(c; x) = u(c) + \delta V[f(x - c)]. \tag{4b}$$

Lemma 2.1

The value function V is continuous and strictly increasing.

Proof. The two properties follow respectively from the theorem of the maximum (Berge, 1959) and the principle of optimality (Bertsekas, 1976). □

Hence, the sup in (4) is achieved, by Weirstrass' theorem, and an optimal stationary policy, to be denoted $g(\cdot)$, exists‡ (cf. Proposition 2, p. 229, of Bertsekas, 1976).

Let the optimal evolution of capital stocks be described by§

$$H(x) = f(x - g(x)), \quad x \geqslant 0. \tag{5}$$

We are ultimately seeking to characterize the properties of the functions V, g and H with $g(\cdot)$ being the global parametric maximizer of $M(c; x)$. However, it turns out that a necessary intermediate step is to study the properties of all the extrema of $M(c; x)$: the local maxima as well as the local minima. That $M(c; x)$ does not necessarily have a unique extremum follows from the fact that V and f are not concave functions (see the example in next section and the discussion following the Main Theorem). Let us now formally define the various notions of extrema:

Definitions

(i) For a fixed $x > 0$, $h(x)$ is a local maximizer (minimizer) of $M(c; x)$ if there exists $\epsilon > 0$ such that $M(h(x), x) \geqslant M(c, x)$ ($\leqslant$) for all c satisfying $0 \leqslant c \leqslant x$ and $|c - h(x)| < \epsilon$.

(ii) For a fixed $x > 0$, $g(x)$ is a global maximizer of $M(c; x)$ if $M(g(x), x) \geqslant M(c, x)$ for all c satisfying $0 \leqslant c \leqslant x$.

(iii) For a fixed $x > 0$, denote by $X(x)$ and $N(x)$ the sets of local maximizers and local minimizers, respectively. The set of extrema of $M(c; x)$ is then given by $E(x) = X(x) \cup N(x)$.

(iv) With I generally referring to compact convex subsets of S, $h \in E(I)$ means $h(x) \in E(x), \forall x \in I$. A similar meaning is attached to $X(I)$ and $N(I)$. The domain I_h of an extremizer h is the largest I for which h is continuous.¶

(v) Let $V_h(x)$ and $H_h(x)$ be the local value function and growth function, respectively, corresponding to $h(x) \in E(x)$, i.e.

$$V_h(x) = u[h(x)] + \delta V[f(x - h(x))] \tag{6}$$

and

$$H_h(x) = f(x - h(x)). \tag{7}$$

It follows from these definitions that the solution to the functional equation (4), V, is the upper envelope of the local value functions, i.e.

$$V(x) = \sup_{h \in E(x)} V_h(x), \quad x \geqslant 0 \tag{8}$$

and g is the optimal policy (not necessarily single-valued) associated with V.

Properties of local extremizers whose domain includes an open set follow:

†We take (4) to be the definition of V throughout this paper.

‡This approach to the existence question is an easy alternative to the usual arguments in sequence spaces.

§Both g and H are set-valued functions, so that $H(x)$ and $g(x)$, in (5), are corresponding selections from H and g, at x.

¶For some h, I_m may consist of a single point, since local extremizers are only u.s.c., by Berge's theorem. Corollary 2.3 sheds some light on this issue.

Lemma 2.2

A necessary condition for h to be in $X(I)(N(I))$ is that

$$\frac{h(x)-h(y)}{x-y} \leqslant 1 \quad (\geqslant 1), \quad \forall x, y \in I, x \neq y.$$

Proof. For $h \in X(I)$, $x \in I$ and $\alpha > 0$ small enough, we have (if α is such that $|h(x+\alpha)-h(x)-\alpha| < \epsilon$, where ϵ is as in Definition 2.1)

$$V_h(x) = u[h(x)] + \delta V[H_h(x)], \tag{9a}$$

$$V_h(x) \geqslant u[h(x+\alpha)-\alpha] + \delta V[H_h(x+\alpha)]. \tag{9b}$$

Similarly:

$$V_h(x+\alpha) = u[h(x+\alpha)] + \delta V[H_h(x+\alpha)], \tag{10a}$$

$$V_h(x+\alpha) \geqslant u[h(x)+\alpha] + \delta V[H_h(x)]. \tag{10b}$$

Adding up (9) and (10) yields

$$u[h(x)] + u[h(x+\alpha)] \geqslant u[h(x+\alpha)-\alpha] + u[h(x)+\alpha] \tag{11a}$$

or equivalently,

$$u[h(x+\alpha)] - u[h(x+\alpha)-\alpha] \geqslant u[h(x)+\alpha] - u[h(x)]. \tag{11b}$$

By the strict concavity of u and the fact that the two arguments in the LHS of (11b) and those in the RHS both differ by α, it follows that

$$h(x+\alpha)-\alpha \leqslant h(x) \quad \text{or} \quad \frac{h(x+\alpha)-h(x)}{\alpha} \leqslant 1. \tag{12a}$$

Since (12a) holds for all $x \in I$ and all α sufficiently small, we conclude that (by integration)

$$\frac{h(x)-h(y)}{x-y} \leqslant 1, \quad \forall x, y \in I. \tag{12b}$$

Now, for $h \in N(I)$, the above argument may be repeated by replacing "$\leqslant$" by "$\geqslant$" and vice-versa, to get that local minima satisfy

$$\frac{h(x)-h(y)}{x-y} \geqslant 1, \quad \forall x, y \in I. \ \square \tag{13}$$

A number of important corollaries follow:

Corollary 2.3

Any extremizer $h \in E(I)$ is a function of bounded variation on I.

Proof. Observe first that Lemma 2.2 is equivalent to:

$$H_h(x) = f(x-h(x)), \quad x \in I$$

is an increasing (decreasing) function if $h \in X(I)(N(I))$. Hence $h(x) = x - f^{-1}[H(x)]$ is increasing (the difference of two increasing functions) is $h \in N(I)(X(I))$. So h is of bounded variation on I (Royden, 1968). $\square$

Consequently, $h(x^+)$ and $h(x^-)$, the left and right limits of h at x, exist for all $x \in I$ and $h'(x)$ exists a.e. in I (Royden, 1968).

Corollary 2.4

The local value function V_h, corresponding to the extremizer $h \in E(I)$, is continuously differentiable on I and $V'(x) = u'[h(x)]$, $x \in I$.

Proof. Subtracting (9a) from (10b) and (9b) from (10a) yields, respectively (for $h \in X(I)$)

$$V_h(x+\alpha) - V_h(x) \geqslant u[h(x)-\alpha] - u[h(x)]$$

and

$$V_h(x+\alpha) - V_h(x) \leqslant u[h(x+\alpha)] - u[h(x+\alpha) - \alpha].$$

Hence

$$\frac{u[h(x)+\alpha] - u[h(x)]}{\alpha} \leqslant \frac{V_h(x+\alpha) - V_h(x)}{\alpha} \leqslant \frac{u[h(x+\alpha)] - u[h(x+\alpha)-\alpha]}{\alpha}. \tag{14}$$

Taking the limit as $\alpha \downarrow 0$ and invoking Corollary 2.3:

$$V_h'(x^+) = u'[h(x)].$$

A similar manipulation starting at $x - \alpha$ would yield

$$V_h'(x^-) = u'[h(x)]$$

and hence V' is continuous at x and

$$V'(x) = u'[h(x)], \quad \forall x \in I. \tag{15}$$

Finally, the above proof may be repeated for $h \in N(I)$ by replacing every "$\geqslant$" by "$\leqslant$" and vice versa. □

3. CHARACTERIZATION OF GLOBAL OPTIMALITY

This section contains the Main Theorem which gives necessary and sufficient conditions for a consumption policy g to be the global maximizer for the optimization problem given by (1)–(3).

We start by extending the properties described in the corollaries of the previous section to g.

Lemma 3.1

A necessary condition for g to be the global maximizer for the optimization problem (1)–(3) is that

$$\frac{g(x) - g(y)}{x - y} \leqslant 1$$

for all distinct x, y in $[0, \max\{x_0, \bar{x}\}]$.

Proof. Simply replace V_h by V and h by g in the proof of Lemma 2.2. □

Remark. Since V is the upper envelope of the V_hs and g is the corresponding maximizer, $V_{h_1}(x) = V_{h_2}(x) = V(x)$ for some x implies that $g(x) = \{h_1(x), h_2(x)\}$. Furthermore, the possibility $g(x) = [h_1(x), h_2(x)]$ is not ruled out, since it preserves the upper-semi-continuity of g (Berge, 1959) and the condition given in Lemma 3.1.

Let s denote any single-valued selection from the set-valued function g. Note that as a consequence of Lemma 3.1 and the above remark, g always admits unique upper- and lower-semi-continuous (u.s.c. and l.s.c.) selections, but may fail to have any nonsemi-continuous as well as any continuous selections.

Corollary 3.2

Any selection s from the optimal consumption policy, is of bounded variation on $[0, \max\{x_0, \bar{x}\}]$.

Proof. Clearly, s satisfies

$$\frac{s(x) - s(y)}{x - y} \leqslant 1.$$

See the proof of Corollary 2.3. □

Corollary 3.3

The left and right derivatives of the value function V exist for all x in $[0, \max\{x_0, \bar{x}\}]$ and satisfy

$$V'(x^-) = u'[g(x^-)] \leqslant V'(x^+) = u'[g(x^+)].$$

Proof. This proof may be found (in an equivalent form) in Dechert and Nishimura, or in our setting in Amir (1984) or Amir *et al.* (1984) but is given here for completeness. Let s_u and s_l denote the u.s.c. and l.s.c. selections from g, respectively. Repeating the argument in the proof of Lemma 2.2 with V_h replaced by V and h by s_1, we arrive at the analog of equation (14):

$$\frac{u[s_1(x)+\alpha]-u[s_1(x)]}{\alpha} \leqslant \frac{V(x+\alpha)-V(x)}{\alpha} \leqslant \frac{u[s_1(x+\alpha)]-u[s_1(x+\alpha)-\alpha]}{\alpha}.$$

Taking the limit as $\alpha \downarrow 0$ and invoking Corollaries 3.2 and 3.1, it follows that $V'(x^+) = u'[s_1(x)]$.

Similarly, starting at the stock level $x-\alpha$ and selecting s_u, we get

$$V'(x^-) = u'[s_u(x)].$$

Clearly, $s_1(x) = g(x^+)$ and $s_u(x) = g(x^-)$. Moreover, by Lemma 3.1 $g(x^+) \leqslant g(x^-)$, so that $u'[g(x^+)] \geqslant u'[g(x^-)]$. □

In view of Corollary 2.4, V may be regarded as the pointwise supremum of a collection of differentiable functions. The set of points at which V is not differentiable coincides with the set of points at which g is not single-valued. It is a countable set, by Corollary 3.2 (a function of bounded variation, being the difference between two monotone increasing functions, has at most countably many points of discontinuity, all of the first kind). Nevertheless, when restricted to the range of H, V is differentiable.

Lemma 3.4

The value function V is continuously differentiable at $H(x)$ for all x in $[0, \max\{x_0, x\}]$ and $V'[H(x)] = u'\{g[H(x)]\}$.

Proof. Suppose there exists x_0 such that V is not differentiable at $H(x_0)$. By Corollary 3.3, $V'[H(x_0)^-]$ and $V'[H(x_0)^+]$ exist. Therefore

$$\frac{\partial M[g(x_0)^-, x_0]}{\partial c} > 0 > \frac{\partial M[g(x_0)^+, x_0]}{\partial c}.$$

Note that only one of the above inequalities need be strict. It follows that

$$u'[g(x_0)] - \partial V'[H(x_0)^+]f'(x-g(x_0)) > u'[g(x_0)] - \delta V'[H(x_0)^-]f'(x-g(x_0))$$

or

$$V'[H(x_0)^+] < V'[H(x_0)^-],$$

a contradiction to Corollary 3.3.

Since g is continuous at $H(x)$ and $V'[H(x)] = u'\{g[H(x)]\}$, V' is continuous at $H(x)$. □

Another way to state Lemma 3.4 is: H and g are continuous at any point x such that there exists y with $x = H(y)$. The equivalence of these two statements is clear from Corollary 3.3 and the remarks following its proof. All the points of discontinuity of g and H are thus such that they cannot equal $H(y)$ for any $y \in S$.

We now turn to the statement of the Main Theorem, the proof of which is in the Appendix.

Main Theorem

A set-valued function g is the global maximizer for the optimization problem given by (1)–(3) if and only if:

(a) $u'[g(x)] = \delta u'\{g[H(x)]\}f'(x-g(x)), \quad \forall x \in S$;

(b) $\dfrac{g(x)-g(y)}{x-y} < 1, \quad \forall x, y \in S, \quad x \neq y$;

(c) V_g, are defined by (6), is a continuous increasing function.

In what follows, we discuss the meaning and implications of each of these four conditions. (a) is simply the Euler equation, obtained by setting $\partial M(c;x)/\partial c = 0$. Thus, we could have defined an extremizer $h(\cdot)$ of $M(c;x)$ as a continuous solution to

$$u'[h(x)] = \delta u'\{g[H_h(x)]\}f'(x-h(x)). \tag{16}$$

The domain I_h of h would then be the set of all x for which the above equation holds.

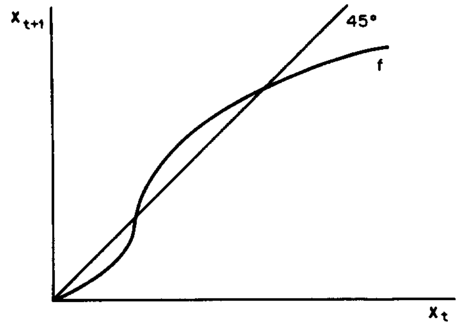

Fig. 1

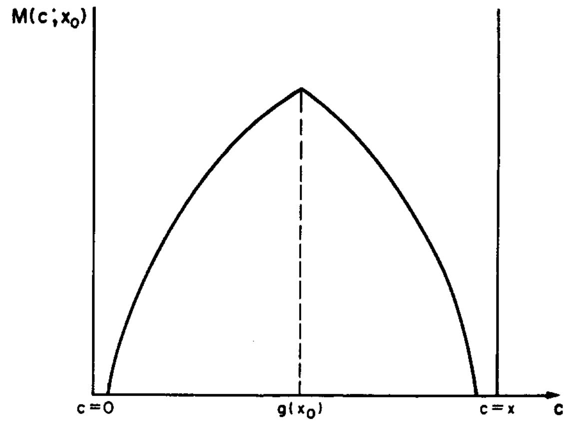

Fig. 2

Condition (b) is readily seen to be equivalent to the monotonicity property of optimal paths in Dechert and Nishimura (1983). Put differently, it says that H is an increasing function. It follows from Lemma 2.2 (where the inequalities are actually strict; see Appendix) that (b) is satisfied by any local maximizer, while local minimizers satisfy the opposite (strict) inequality. In fact, it is shown in the Appendix that $h(x)-h(y)/(x-y)<1$ (>1) for all distinct x, y in I is equivalent to $\partial^2 M[h(x), x]/\partial c^2 < 0$ (>0) for all x in I, the equivalence being up to the fact that the second partial of M w.r.t. c is only known to exist a.e. x (see Appendix).

It appears then that the previous literature in the nonclassical case characterizes "locally optimal paths," and offers no method or approach on how to extricate the "globally optimal paths" in an unequivocal manner. The existence of interior local (but non-global) maximizers is established, in the one-period horizon context, by an example contained in Amir (1984) and Amir *et al.* (1984). A different example achieving the same aim is given in the next section.

We now give an argument based on our results to show existence of interior local minimizers (which are not global): Suppose x_0 is a stock level at which there are two possible optimal paths (e.g. an extinction path and a path of accumulation to the stable steady-state equilibrium). The corresponding $M(c;x_0)$ is depicted in Fig. 3a, with $g(x_0)=\{h_1(x_0), h_2(x_0)\}$. Now consider $M(c;x_0+\epsilon)$ with $\epsilon>0$ small enough. By Lemma 2.1, the graph is continuously deformed, and by Lemma 3.1, $g(x_0+\epsilon)=h_1(x_0+\epsilon)$ [and not $h_2(x_0+\epsilon)$], so that $M(c;x_0+\epsilon)$ is as shown in Fig. 3b. Clearly, both $h_1(x_0+\epsilon)$ and $h_2(x_0+\epsilon)$ satisfy the Euler equation (16). Furthermore, Lemma 2.2 implies that both h_1 and h_2 have all their slopes bounded above by unity. So the question now is: How does one choose between h_1 and h_2 at $x_0+\epsilon$? It turns out that condition (d) provides the answer to this question, as is established in the Appendix.

4. AN EXAMPLE

To illustrate the points made at the end of the previous section, a specific one-period horizon example is given here, using a convex production function. The search for possible examples with longer horizons, involving the same points of interest, is extremely complex.

For a one-period horizon problem, one needs to solve the following:

$$V_1(x)=\max_{0\leqslant c\leqslant x}\{u(c)+\delta u[f(x-c)\}. \tag{17}$$

Observe that if the maximand in (17) has interior local nonglobal maximizers, the same is likely to hold for longer horizons since u gets replaced by $V_1, V_2, \ldots$ which are not concave functions.

For the present example, consider $u(c)=\ln c, c>0$ and $f(x)=e^{2x^2}$, $x\geqslant 0$. Equation (17) becomes

$$V_1(x)=\max_{0\leqslant c\leqslant x}\{\ln c+2(x-c)^2\}. \tag{18}$$

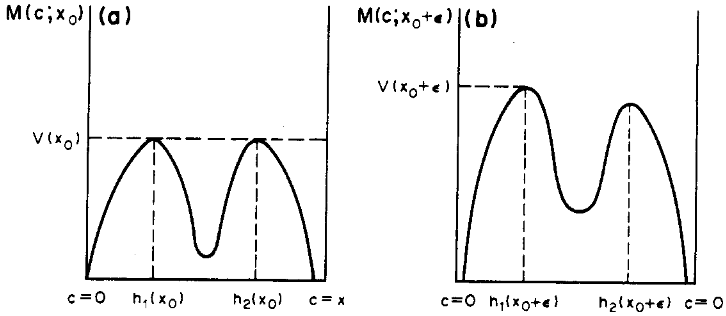

Fig. 3

The three possible configurations of $M_1(c;x)$, the maximand in (18), are depicted in Fig. 4.

The first-order condition for an interior extremum in (18) is $1/c = 4(x-c)$ or

$$4c^2 - 4xc + 1 = 0. \tag{19}$$

The two solutions to this equation (the interior extremizers of M_1) are

$$h_+(x) = \tfrac{1}{2}[x + (x^2-1)^{1/2}] \quad \text{and} \quad h_-(x) = \tfrac{1}{2}[x - (x^2-1)^{1/2}] \quad \text{for } x \geqslant 1.$$

The marginal propensities to consume are given by

$$h'_+(x) = \frac{1}{2}\left(1 + \frac{x}{(x^2-1)^{1/2}}\right) \quad \text{and} \quad h'_-(x) = \frac{1}{2}\left(1 - \frac{x}{(x^2-1)^{1/2}}\right), \quad x \geqslant 1.$$

If $x < 1$, no interior extrema exist and $M_1(c;x)$ is maximized by $h_c(x) = x$. Notice that if we want to illustrate the same points without making use of local corner maximizers, by an example with closed-form solutions, we will have to solve a third degree polynomial instead of (19). (This is because having two interior local maximizers would imply the existence of a local minimizer, since M is continuous in c.)

If $x \geqslant 1$, it can easily be verified that $0 < h_-(x) \leqslant h_+(x) < x$, i.e. both h_- and h_+ are interior and feasible. Furthermore, $h'_-(x) < 0 < 1 < h'_+(x)$, indicating, in view of Lemma 2.2, that h_- is a local maximizer while h_+ is a local minimizer.

Now, let us compare h_- and h_c for $x \geqslant 1$. To h_c corresponds the local value function V_{h_c} given by $V_{h_c}(x) = \ln x$. To h_- corresponds the local value function V_{h_-} given by

$$V_{h_-}(x) = \ln\{\tfrac{1}{2}[x - (x^2-1)^{1/2}]\} + \tfrac{1}{2}[x + (x^2-1)^{1/2}]^2.$$

At $x = 1$, we have

$$V_{h_c}(1) = 0 > V_{h_-}(1) = \tfrac{1}{2} + \ln\tfrac{1}{2}.$$

It can be shown that there exists a unique $y > 1$ with the property that

$$V_{h_c}(y) = V_{h_-}(y) \quad \text{and} \quad V_{h_c}(x) \gtrless V_{h_-}(x) \quad \text{if } x \lessgtr y.$$

Hence, the global maximizer of $M_1(c;x)$ is given by the upper-hemi-continuous correspondence

$$g(x) = \begin{cases} x, & x \leqslant y \\ \tfrac{1}{2}[x - (x^2-1)^{1/2}], & x \geqslant y. \end{cases}$$

The value function is given by the continuous function

$$V(x) = \begin{cases} \ln x, & x \leqslant y \\ \ln\{\tfrac{1}{2}[x - (x^2-1)^{1/2}]\} + \tfrac{1}{2}[x + (x^2-1)^{1/2}]^2, & x \geqslant y. \end{cases}$$

Consumption in the second period is given by the upper-hemi-continuous correspondence:

$$H(x) = \begin{cases} 0, & x \leqslant y. \\ \exp\{\tfrac{1}{2}[x + (x^2-1)^{1/2}]^2\}, & x \geqslant y. \end{cases}$$

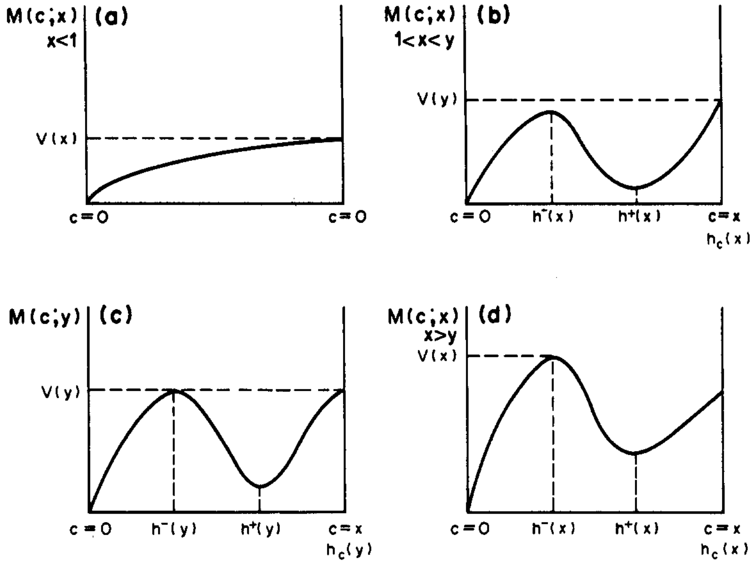
M(c;x) (a)
x<1
V(x)
c=0
c=0
M(c;x) (b)
1<x<y
V(y)
c=0
h⁻(x)
h⁺(x)
c=x
h_c(x)
M(c;y) (c)
V(y)
c=0
h⁻(y)
h⁺(y)
c=x
h_c(y)
M(c;x) (d)
x>y
V(x)
c=0
h⁻(x)
h⁺(x)
c=x
h_c(x)

Fig. 4

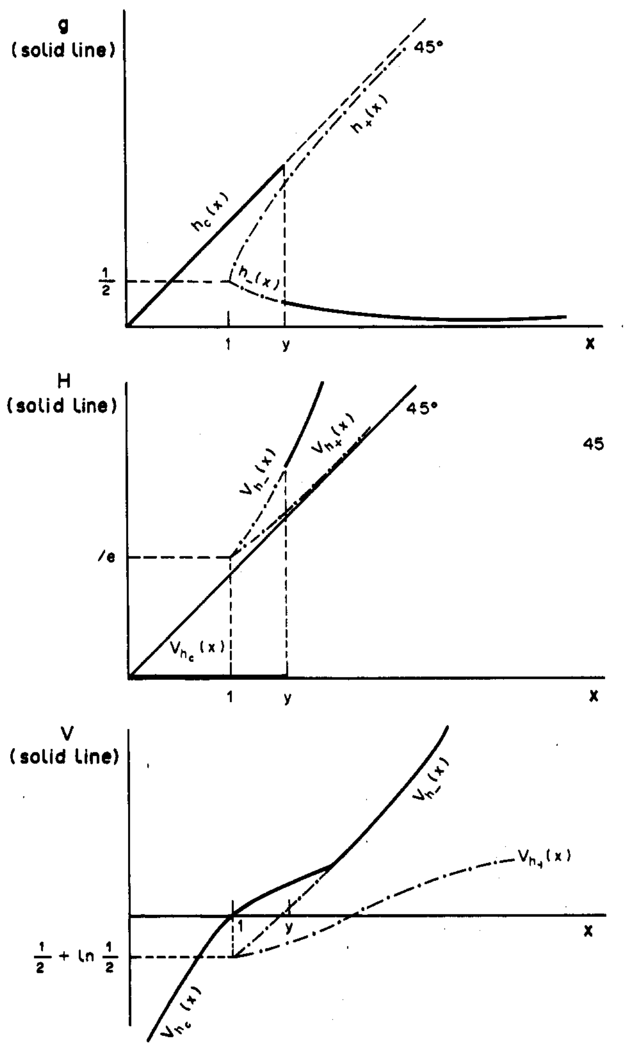
g
(solid line)
45°
h₊(x)
h_c(x)
h₋(x)
1/2
1
y
x
H
(solid line)
45°
45
V_{h₊}(x)
V_{h₋}(x)
/e
V_{h_c}(x)
1
y
x
V
(solid line)
V_{h₋}(x)
V_{h₊}(x)
1
y
x
1/2 + ln 1/2
V_{h_c}(x)

Fig. 5

In all the above expression, y is the solution of the equation

$$\log x = \ln\{\tfrac{1}{2}[x - (x^2 - 1)^{1/2}]\} + \tfrac{1}{2}[x + (x^2 - 1)^{1/2}]^2.$$

See Fig. 5 for graphs, and compare with Fig. 4.

5. ASYMPTOTIC PROPERTIES OF GLOBALLY OPTIMAL PATHS AND CONCLUSION

The asymptotic properties of optimal paths are given in Dechert and Nishimura (1983), using the Euler equation and the monotonicity property of optimal paths (or, equivalently, the strict boundedness of the marginal properties of consumption by unity, i.e. Lemma 3.1 plus Lemma A.1 here). It turns out that these properties depend essentially on two factors: (i) Whether H starts above or below the 45° degree line (i.e. whether $\delta f'(0) \geqslant 1$ or $\delta f'(0) < 1$), respectively, (ii) the number of steady-state equilibria (i.e. fixed-points $\tilde{x}$ of H). $\tilde{x}$ must satisfy $\tilde{x} = f(\tilde{x} - g(\tilde{x}))$ and $\delta f'(\tilde{x} - g(\tilde{x})) = 1$, whence $\tilde{x} = f[f'^{-1}(1/\delta)]$. There are either 0, 1, or 2 fixed points of H, the location of which only depends on f.

If $\delta f'(0) \geqslant 1$, H must be a continuous function (this is a consequence of Lemma 3.4 and the remarks following its proof), and have one globally stable fixed point at $f[(f'^{-1}(1/\delta)]$.

If $\delta f'(0) \leqslant 1$ and there exists $x > 0$ with $H(x) > x$, H will have one stable fixed point $\tilde{x}$ and either an unstable fixed point (in which case H is also continuous) or one (or more) jump discontinuities (all to the left of $\tilde{x}$). In the latter case (the most interesting one), assume that at some stock level one of the two mistakes, described in the two paragraphs before last of the Appendix, was committed in selecting the global maximizer g. Then, the resulting growth functions H_h would have the same asymptotic properties as the true H, but a discontinuity at a different point (see Fig. 5 for some such examples). Observe that in this case, the resulting optimal paths would not coincide with the true optimal paths, and that, in particular, Clark's (1971) minimum safe standard of conservation would be incorrectly located.

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APPENDIX

Proof of the Main Theorem

We prove each of the four conditions separately, and include additional comments pertaining to the meaning of each condition. As some of the arguments are rather long and intricate, we break them into intermediate lemmae.

Proof of (a). This is the Euler equation or first-order necessary condition for the maximization in (4). By Lemma 3.4, this condition is, for all x in $[0, \max\{x_0, \tilde{x}\}]$:

$$u'[g(x)] = \delta V'[H(x)]f'(x - g(x)) = \delta u'\{g[H(x)]\}f'(x - g(x)). \ \square$$

Proof of (b). This is a second-order necessary condition, which is sufficient for local optimality, but not for global optimality, as will become clear from the following arguments. We first prove that it is a necessary condition. To this end, we need:

Lemma A.1

H is an injective function (i.e. $H(x) = H(y)$ implies $x = y$).

Proof. If $H(x) = H(y)$, then $x - g(x) = y = g(y)$, so that the RHS of the Euler equation takes the same value at x and at y. Hence, $u'[g(x)] = u'[g(y)]$ or $g(x) = g(y)$, so that $x = y$. □

In view of Lemma 3.1, to prove necessity of (c), it only remains to show that $g(x) - g(y) \neq x - y$ for all distinct x and y. But this is precisely equivalent to $H(x) \neq H(y)$ for all distinct x and y, i.e. Lemma A.1. Note that a similar argument would show that the inequalities in Lemma 2.2 (for local extremizers) are also strict.

We now show that (c) is sufficient for local optimality. Let S be the set of points at which s' (s being any selection from g) exists, and $\bar{S}$ its complement in $[0, \max\{(x_0, \tilde{x}\}]$. By Corollary 3.2, $\bar{S}$ is of measure zero (Royden, 1968).

If $x \in S$, then condition (c) implies that $g'(x) \leqslant 1$. We first prove that this inequality is actually strict. Suppose that for some $x_0 \in S, g'(x_0) = 1$. Differentiating the Euler equation (for x in S) yields

$$u''[g(X)]g'(x) = \delta\{V''[H(x)]f'^2(x - g(x)) + V'[H(x)]f''(x - g(x))\}(1 - g'(x)). \tag{A.1}$$

At x_0, the LHS of (A.1) is equal to $u''[g(x_0)]$ and the RHS vanishes, unless $V''[H(x_0)] = -\infty(V'[H(x_0)]$ is finite by Lemma 3.4). Since $V''[H(x_0)] = u''\{g[H(x_0)]\}g'[H(x_0)]$, this implies that $g'[H(x_0)] = +\infty$, a contradiction to Lemma 3.1. We conclude that $g'(x) < 1, \forall x \in S$.

To establish sufficiency for local optimality, observe that for $x \in S$, $g'(x) < 1$ is the same as

$$u''[g(x)] + \frac{u''[g(x)]g'(x)}{1 - g'(x)} < 0,$$

which, in view of equation (1), is equivalent to (for $x \in S$)

$$u''[g(x)] + \delta\{V''[H(x)]f'^2(x - g(x)) + V'[H(x)]f''(x - g(x))\} < 0.$$

But this is precisely (for $x \in S$, i.e. for almost all x):

$$\frac{\partial^2 M[g(x), x]}{\partial c^2} < 0.$$

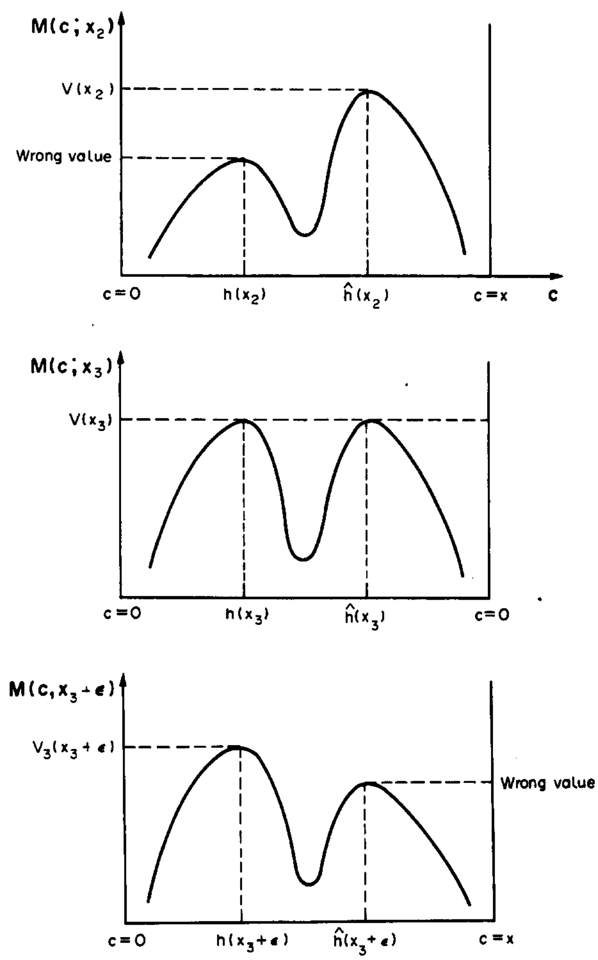

Fig. A.1

If $x \in \bar{S}$, $g'(x)$ does not exist and thus $\partial^2 M[g(x), x]/\partial c^2$ does not exist either (note that, by equation (1), the two exist at the same xs). For $x \in \bar{S}$, consider the four Dini derivates of g at x. These are the lim sup and lim inf of the directional incrementary ratios of g, and thus always exist in the extended reals (see Titchmarsh, 1938). Since the incrementary ratios (slopes) of g are bounded above by one, so are the Dini derivates of g (Titchmarsh, 1938). Repeat the above argument for local sufficiency for $x \in \bar{S}$ by replacing derivatives by each of the four Dini derivates to conclude first that the four Dini derivates of g at $x \in \bar{S}$ are strictly less and one, then those of $\partial M[g(x), x]/\partial c$ w.r.t. c are strictly negative. Hence, M has a local maximum at x and not an inflection point. □

Proof of (c). The (d) is a necessary condition is the content of Lemma 2.1. Here, we establish that (d) is also sufficient for global optimality, through a series of intermediate lemmae.

The idea behind the overall proof is that, if at some stock level x, a local nonglobal maximizer has been selected as $g(x)$, then a downward jump discontinuity would appear in the resulting value function at x or at some point to the right of x.

We first establish some properties of g near the origin.

Lemma A.2

Any selection s from the optimal consumption policy g is continuous and increasing in a neighborhood of the origin.

Proof. From $0 \leqslant s(x) \leqslant x$, it follows that $s(0) = 0$ and s is continuous at 0 (s, being of bounded variation, can only have discontinuities of the first kind, i.e. finite jumps, but such a jump would violate interiority at 0). We now show that s is continuous in a neighborhood of 0. To this end, it suffices to show that 0 is not a limit point of a sequence of points of jump discontinuity of s, But, by Lemma 3.1, all such jumps must be downward (i.e. such that $s(x^-) > s(x^+)$), a violation of interiority.

Since s is continuous near 0, and is interior, it is clearly increasing near 0. □

Lemma A.3

There exists a neighborhood N of the origin such that $E(N) = \{g\}$.

Proof. Assume on the contrary that any neighborhood M of 0 is such that $E(M)$ contains another element h (in addition to g). If h is a local minimizer, then by Lemma 2.2 and the remark following the proof of (c), we must have $h(x) - h(y) > x - y$, for all distinct x and y in M. Therefore h is not an interior minimizer of $M(c; x)$, Now, if h is a local maximizer, they by the continuity of M, there exists a local minimizer whose value is between those of h and g. But the above argument implies that this third extremizer is not interior, a contradiction. □

Remark. A continuum of global maximizers sufficiently near 0 is ruled out by Lemma A.2. A continuum of local maximizers at a point x_0 near 0, of the form $[h_1(x_0), h_2(x_0)]$, is ruled out by the argument in the proof of Lemma A.3 in the following manner: Take $\epsilon > 0$ sufficiently small. To each $h(x_0)$ in $[h_1(x_0), h_2(x_0)]$ corresponds as instantaneous rate of change of local value of $V_h'(x_0) = u'[(x_0)]$. Hence, at $x_0 + \epsilon$, only $h_1(x_0)$ will still be a local maximizer. Now, between $h_1(x_0)$ and $g(x_0)$, there must exist a local minimizer. Then, apply the argument in the proof of Lemma A.3 to conclude that this minimizer cannot be interior.

We have also just shown that continuums of extremizers (local or even global) can only occur at countably many points.

The next result holds, similarly, that for sufficiently large values of the stock level, M only has one extremum.

Lemma A.4

There exists $\tilde{x} > 0$ such that $x > \tilde{x}$ implies $E(x) = \{g(x)\}$.

Proof. $\tilde{x}$ here is a point beyond which V and $M(\cdot; x)$ is a concave function, as in the classical case (for a proof, see Mirman, 1980). Hence, for $x > \tilde{x}$, $M(c; x)$ is concave in c, whence the conclusion. □

To recapitulate, Lemmas A.3 and A.4 assure us that, sufficiently near 0 and far enough from 0, there is no possibility of selecting a local non-global maximizer as g. It may thus be said that for such values of x the Euler condition is sufficient for optimality. The remaining part of the proof of (d) takes care of the intermediate values of the stock level x, as follows.

Keeping in mind that the curve $M(c; x)$ moves continuously in x, and that any extremum has a continuous and increasing local value (at the rate $u'[h(x)]$), let x_1 and x_3 be as follows:

$$x_1 = \inf\{x: \text{Card}\, E(x) > 1\}, \quad x_3 = \inf\{x: \text{Card}\, g(x) > 1\}\dagger$$

Clearly, $x_1 < x_3$.‡ Let x_2 be such that $x_1 < x_2 < x_3$. If at x_3 (see Fig. A1), $h(x_2) \neq g(x_2) = \hat{h}(x_2)$ is selected as the global maximizer, the value function V would be discontinuous at x_2 (i.e. $V(x_2^-) > V(x_2^+)$), regardless of whether $h(x_2) > g(x_2)$ or $h(x_2) < g(x_2)$. Hence, we would know that $h(x_2)$ is not the global maximizer at x_2.

Now, at x_3, $g(x_3) = \{h(x_3), \hat{h}(x_3)\}$, and, by Lemma 3.1, $g(x_3 + \epsilon) = h(x_3 + \epsilon) > \hat{h}(x_3 + \epsilon)$, for all small $\epsilon > 0$. Suppose that, at x_3 and $x_3 + \epsilon$, $\hat{h}$ is kept as the global maximizer. The resulting value will remain continuous at x_3 and $x_3 + \epsilon$, with $V_h(x_3 + \epsilon) > V_{\hat{h}}(x_3 + \alpha)$ and the gap $V_h(x) - V_{\hat{h}}(x)$ (i.e. the error in value) will be an increasing function of x, in view of Corollary 2.4. Hence $\hat{h}(x)$ cannot approach $h(x)$ and coincide with it (if it could, and did, say at a point x_4, then we would not know that we had the wrong maximizer between x_3 and x_4). Moreover jumping from $\hat{h}(x)$ to $h(x)$ would yield a downward jump in the (wrong) value. Finally, since, by Lemma A.4, we know that we will eventually pick the right maximizer, an upward jump will result in the (wrong) value, at some point, to get back to $g(x)$.

We have thus proved that only if the right selection of the global maximizer is made at every stock level will the resulting value function be continuous and increasing. □

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†Here Card stands for the cardinality of a set. If such x_1 and x_3 do not exist, $E(x)$ is a singleton, g a single-valued function, and there is nothing to prove.

‡It may be that $x_1 = x_3$, in which case there is a continuum of global maximizers at $x_1 = x_3$, each of which may be selected. Since we are interested in the possibility of wrong selections, the case $x_1 < x_3$ is of interest.